

# Zero Lelong number problem

Alexander Rashkovskii

## Abstract

We discuss several related problems on residual Monge-Ampère masses of plurisubharmonic functions. The note is based on the author's talk at the 27th Congress of Nordic Mathematicians, March 19, 2016.

*Mathematic Subject Classification:* 32U05, 32U25, 32U35, 32W20

## 1 Introduction

Here we give a detailed exposition of Questions 7 and 8 from a recent list of open problems in pluripotential theory [9]. The note is based on the author's talk at the 27th Congress of Nordic Mathematicians, March 19, 2016.

Recall that a function  $u$  in a domain  $\omega \subset \mathbb{C}^n$  is plurisubharmonic if it is upper semi-continuous and such that the composition  $u \circ \gamma$  is subharmonic in the unit disk  $\mathbb{D}$  for any holomorphic mapping  $\gamma : \mathbb{D} \rightarrow \omega$ . By  $\text{PSH}(\omega)$  (resp.,  $\text{PSH}_0$ ) we denote the collection of all plurisubharmonic functions in  $\omega$  (resp., germs of plurisubharmonic functions at  $0 \in \mathbb{C}^n$ ).

A germ  $u \in \text{PSH}_0$  is said to be *singular* if  $u(0) = -\infty$ .

A basic characteristic of singularity of  $u$  is its *Lelong number*

$$\nu_u = \nu_u(0) = \liminf_{z \rightarrow 0} \frac{u(z)}{\log |z|};$$

this is the largest number  $\nu \geq 0$  such that

$$u(z) \leq \nu \log |z| + O(1) \tag{1}$$

near 0. Equivalently,

$$\nu_u = dd^c u \wedge (dd^c \log |z|)^{n-1}(0), \tag{2}$$

where  $d = \partial + \bar{\partial}$ ,  $d^c = (\partial - \bar{\partial})/2\pi i$ .

If  $f \in \mathcal{O}_0$  is a holomorphic function near 0, then  $\nu_{\log |f|} = \text{mult}_0 f$ , the multiplicity of the  $f$  at 0. However, if  $f = (f_1, \dots, f_m)$  is a holomorphic mapping, then  $\nu_{\log |f|} = \min_k \text{mult}_0 f_k$  is far from the multiplicity of the zero.

Let  $\text{PSH}_0^*$  be the germs that are locally bounded outside 0 (i.e., with *isolated singularity* at 0). The complex Monge-Ampère operator  $(dd^c u)^n$  is well defined on such a function  $u$  [6], and we denote by

$$\tau_u = (dd^c u)^n(0)$$

its *residual Monge-Ampère mass* at 0.

For the mappings  $f$  with isolated zero, we have  $\text{mult}_0 f = \tau_{\log |f|}$  [6].

## 2 Relations between the characteristics of singularity

For a holomorphic mapping  $f$  to  $\mathbb{C}^n$ , by the local Bézout's theorem,

$$\text{mult}_0 f \geq \prod_k \text{mult}_0 f_k \geq (\min_k \text{mult}_0 f_k)^n.$$

**Theorem 2.1** [6] *If  $u \in \text{PSH}_0^*$ , then  $\tau_u \geq \nu_u^n$ .*

*Proof.* This follows from relations (1)-(2) and Demailly's Comparison Theorem [6]

$$(CT) \quad u_j \leq v_j + O(1) \Rightarrow \bigwedge_j dd^c u_j(0) \geq \bigwedge_j dd^c v_j(0)$$

applied to  $u_j = u$ ,  $v_j = \nu_u \log |z|$ . □

No reverse bound is possible: take  $u = \max\{k \log |z_1|, \log |z_2|\}$ , then  $\nu_u = 1$  while  $\tau_u = k$ .

**Problem 1** (*Zero Lelong Number Problem*; V. Guedj, A. Rashkovskii, 1999): *Is the implication*

$$(P1) \quad \nu_u = 0 \Rightarrow \tau_u = 0$$

*true whenever  $(dd^c u)^n$  is well defined (e.g., for  $u \in \text{PSH}_0^*$ )?* (This is Question 7 from [9].)

Note that even in  $n = 2$  the condition  $\nu_u = 0$  does not imply  $dd^c u \wedge dd^c v(0) = 0$  for any  $v$ . For example, if  $u = \max\{-|\log |z_1||^{1/2}, \log |z_2|\}$  and  $v = \log |z_1|$ , then  $\nu_u = 0$  and  $dd^c u \wedge dd^c v = \delta_0$ .

## 3 Finite Łojasiewicz exponent

Denote

$$\gamma_u = \limsup_{z \rightarrow 0} \frac{u(z)}{\log |z|},$$

*the Łojasiewicz exponent of  $u$  at 0.*

By (CT), we get

**Theorem 3.1** *Finite Łojasiewicz exponent*

$$(FLE) \quad \gamma_u < \infty$$

*implies (P1) is true.*

**Examples:**

1. Any analytic singularity  $u = \log |f| + O(1) \in \text{PSH}_0^*$  has FLE (the classical Łojasiewicz exponent of the mapping  $f$ ).
2. Multicircled singularities  $u(z) = u(|z_1|, \dots, |z_n|) + O(1) \in \text{PSH}_0^*$  have FLE [12], [13].

## 4 Greenifications

FLE condition might look too restrictive. But, for  $n = 1$ , any (pluri)subharmonic germ  $u$  at 0 represents as  $u(z) = g_u + v(z)$ , where  $g_u(z) = \nu_u \log |z|$  and  $v$  is a (pluri)subharmonic function with zero Lelong number, so  $\nu_{g_u} = \nu_u$  and  $g_u$  definitely has FLE.

For  $n \geq 1$ , a 'greenification' [15] of  $u$  is defined in a neighborhood  $\omega$  of 0 as

$$g_u(z) := \limsup_{x \rightarrow z} \sup \{v(x) : v \in \text{PSH}(\omega), v \leq 0, v \leq u + O(1) \text{ near } 0\}.$$

If  $u \in \text{PSH}_0^*$ , then  $(dd^c g_u)^n = 0$  on  $\omega \setminus 0$ ; in other words,  $g_u$  is a *maximal singularity*. In addition,  $\nu_{g_u} = \nu_u$  and  $\tau_{g_u} = \tau_u$ .

**Theorem 4.1** [18], [15] *For  $u \in \text{PSH}_0^*$ ,  $g_u \equiv 0$  if and only if  $\tau_u = 0$ .*

So: (P1) is equivalent to the following question: *For  $u \in \text{PSH}_0^*$ , does  $\nu_u = 0$  imply  $g_u \equiv 0$ ?*

**Problem 1.1:** *Construct a maximal singularity  $\varphi \in \text{PSH}_0^*$  with  $\gamma_\varphi = \infty$ .*

*Remark:* There exist  $u \in \text{PSH}(\omega)$  with non-isolated, maximal singularity at 0 and well-defined Monge-Ampère operator  $(dd^c u)^n$  [16]; moreover, such a function can have the set  $\{u = -\infty\}$  dense in  $\omega$  [1].

By removing the condition of isolated singularity, one gets

**Problem 1.2.** *For  $u \in \text{PSH}_0$ , is the implication*

$$(P1') \quad \nu_u = 0 \Rightarrow g_u \equiv 0$$

*true?*

More classes with FLE property to come when considering Problem 2 below.

## 5 Intermediate Lelong numbers

For  $u \in \text{PSH}_0^*$ , denote

$$e_k = e_k(u) = (dd^c u)^k \wedge (dd^c \log |z|)^{n-k}(0), \quad k = 0, \dots, n,$$

so  $e_0 = 1$ ,  $e_1 = \nu_u$ ,  $e_n = \tau_u$ . If  $u = \log |f|$  for a holomorphic mapping  $f$  with the zero set of codimension  $k$ , then  $e_k$  equals its multiplicity at 0.

As follows from [4], these *intermediate Lelong numbers* satisfy

$$e_k^2 \leq e_{k-1} \cdot e_{k+1}$$

which was noticed in [8] (in analytic setting  $u = \log |f|$ , it was established in [17]).

**Theorem 5.1** [8]  *$e_1 = 0$  implies  $e_k = 0$  for all positive  $k < n$ .*

## 6 Demailly's approximations

Problem 1 may be approached by approximating  $u$  by functions for which the (affirmative) answer to Problem 1 is known: e.g., by those with analytic singularities.

We recall a procedure for analytic approximations due to Demailly [5]. Given  $u \in \text{PSH}(\omega)$  and  $k \in \mathbb{Z}_+$ , let  $\{f_{k,m}\}_m$  be an orthonormal basis of the weighted Hilbert space

$$H_k(u) = \{f \in \mathcal{O}(\omega) : \int_{\omega} |f|^2 e^{-2ku} dV < \infty\}.$$

Then the functions

$$\mathcal{D}_k u = \frac{1}{2k} \log \sum_m |f_{k,m}|^2 \in \text{PSH}(\omega)$$

satisfy

$$u \leq \mathcal{D}_k u + \frac{C}{k}$$

and converge to  $u$  as  $k \rightarrow \infty$  (in  $L^1_{loc}$  and pointwise). Moreover,  $\nu_{\mathcal{D}_k u} \rightarrow \nu_u$ .

Assume  $u \in \text{PSH}_0^*$ , then  $\mathcal{D}_k u \in \text{PSH}_0^*$  have analytic singularities. The condition  $\nu_u = 0$  implies  $\nu_{\mathcal{D}_k u} = 0$ , which in turn, since  $\mathcal{D}_k u$  have analytic singularities, gives us  $\tau_{\mathcal{D}_k u} = 0$ , and it remains to relate these to  $\tau_u$ .

**Problem 2** (*Demailly*): *Is the convergence*

$$(P2) \quad \tau_{\mathcal{D}_k u} \rightarrow \tau_u$$

true? (Question 8 from [9].)

It is known [2] that the functions

$$\mathcal{D}_{2^k} u + \frac{C}{2^{k+1}}$$

decrease to  $u$  and so, by [6],  $(dd^c \mathcal{D}_{2^k} u)^n$  converge to  $(dd^c u)^n$  as measures (which does not guarantee convergence of their masses at 0).

## 7 When (P2) is true

For some classes of functions, convergence (P2) is known. Namely, this is so for:

1. *Analytic singularities* [3]  $u = c \log |f| + O(1) \in \text{PSH}_0^*$ , where  $f$  are holomorphic mappings with isolated zero.

Such functions are a particular case of

2. *Exponentially Hölder continuous functions* [3]  $\varphi \in \text{PSH}_0^*$ ,

$$e^{\varphi(x)} - e^{\varphi(y)} \leq A|x - y|^\beta, \quad \beta > 0.$$

A more general class are:

3. *Tame singularities* :  $\varphi \in \text{PSH}_0^*$  with the property that there exists  $C > 0$  such that  $\forall t > C$  and every  $f \in \mathcal{O}_0$  the condition  $|f|e^{-t\varphi} \in L_{loc}^2$  implies  $\log |f| \leq (t - C)\varphi + O(1)$ . These functions are characterized [3] by the inequalities

$$u + O(1) \leq \mathcal{D}_k u \leq (1 - C/k)u + O(1),$$

so (P2) follows from (CT).

And even more generally, the convergence holds for

4. *Asymptotically analytic singularities* [16]:  $\forall \epsilon > 0 \exists \varphi_\epsilon$  with analytic singularities such that

$$(1 + \epsilon)\varphi \leq \varphi_\epsilon \leq (1 - \epsilon)\varphi.$$

In particular, any isolated multicircled singularity is asymptotically analytic [16].

**Theorem 7.1** [16]  $\varphi \in \text{PSH}_0^*$  is asymptotically analytic if and only if the greenifications  $g_{\mathcal{D}_k \varphi}$  satisfy

$$g_{\mathcal{D}_k \varphi} / g_\varphi \rightarrow 1 \tag{3}$$

uniformly on  $\omega \setminus 0$ .

Since the greenifications of analytic singularities are continuous [19], convergence (3) implies  $g_\varphi \in C(\omega)$  for asymptotically analytic  $\varphi$ .

**Problem 2.1:** Construct  $u \in \text{PSH}_0^*$  with discontinuous  $g_u$ .

**Problem 2.2:** Construct  $u \in \text{PSH}_0^*$  whose singularity is not asymptotically analytic.

The type of  $u \in \text{PSH}_0$  relative to a maximal weight  $\varphi \in \text{PSH}_0^*$  [16] is

$$\sigma(u, \varphi) = \liminf_{z \rightarrow 0} \frac{u(z)}{\varphi(z)}.$$

Equivalently, it is the largest  $\sigma \geq 0$  such that  $u(z) \leq \sigma\varphi(z) + O(1)$ .

**Example:**  $\nu_u = \sigma(u, \log |z|)$ . More generally, let  $\phi_a(z) = \max_i a_i^{-1} \log |z_i|$ ,  $a_i > 0$ , then  $\sigma(u, \phi_a)$  is Kiselman's directional Lelong number [11] in the direction  $a = (a_1, \dots, a_n)$ .

It is known that the Lelong numbers of  $\mathcal{D}_k u$ , both classical and directional, converge to those of  $u$  [5], [13], and the same do their log canonical thresholds [7].

**Theorem 7.2** [16] The types  $\sigma(\mathcal{D}_k u, \varphi) \rightarrow \sigma(u, \varphi)$  for any  $u \in \text{PSH}_0$  if and only if  $\varphi$  has asymptotically analytic singularity.

## 8 Functions with (P2) property

**Theorem 8.1** [16] *For  $u \in \text{PSH}_0^*$ , TFAE:*

- (i)  $\sup_k \tau_{\mathcal{D}_k u} = \tau_u$ ;
- (ii)  $\lim_{k \rightarrow \infty} \tau_{\mathcal{D}_k u} = \tau_u$ ;
- (iii)  $\inf_k g_{\mathcal{D}_k u} = g_u$ ;
- (iv)  $\lim_{k \rightarrow \infty} g_{\mathcal{D}_k u} = g_u$ ;
- (v) *there exist analytic singularities  $\varphi_j \geq u$  such that  $\tau_{\varphi_j} \rightarrow \tau_u$ ;*
- (vi) *there exist maximal analytic singularities  $\varphi_j$  decreasing to  $g_u$ ;*
- (vii) *there  $s_k > 0$  and divisorial valuations  $\mathcal{R}_k$ ,  $k = 1, 2, \dots$ , such that*

$$\sigma(v, g_u) = \inf_k s_k \mathcal{R}_k(v) \quad \forall v \in \text{PSH}_0.$$

**Problem 2.3:** *Is (P2) true for  $u$  with FLE?*

## References

- [1] P. ÅHAG, U. CEGRELL, PHAM HOANG HIEP, Monge-Ampere measures on subvarieties, *J. Math. Anal. Appl.* **423** (2015), no. 1, 94–105.
- [2] Z. BŁOCKI, Some applications of the Ohsawa-Takegoshi extension theorem, *Expositiones Mathematicae* **27** (2009), 125–135.
- [3] S. BOUCKSOM, C. FAVRE, AND M. JONSSON, *Valuations and plurisubharmonic singularities*, *Publ. Res. Inst. Math. Sci.* **44** (2008), no. 2, 449–494.
- [4] U. CEGRELL, *The general definition of the complex Monge–Ampère operator*, *Ann. Inst. Fourier (Grenoble)* **54** (2004), no. 1, 159–179.
- [5] J.-P. DEMAILLY, *Regularization of closed positive currents and intersection theory*, *J. Algebraic Geometry* **1** (1992), 361–409.
- [6] J.P. DEMAILLY, *Monge-Ampère operators, Lelong numbers and intersection theory*, *Complex Analysis and Geometry (Univ. Series in Math.)*, ed. by V. Ancona and A. Silva, Plenum Press, New York 1993, 115–193. Available as Ch. 3 of the Open Content Book: J.P. DEMAILLY, *Complex analytic and differential geometry*, <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>.
- [7] J.P. DEMAILLY AND J. KOLLÁR, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, *Ann. Sci. Ecole Norm. Sup. (4)* **34** (2001), no. 4, 525–556.
- [8] J.-P. DEMAILLY AND PHAM HOANG HIEP, *A sharp lower bound for the log canonical threshold*, *Acta Math.* **212** (2014), no. 1, 1–9.
- [9] S. DINEW, V. GUEDJ, A. ZERIAHI, Open problems in pluripotential theory, *Complex Var. Elliptic Equ.* (2016); available at arXiv:1511.00705.
- [10] C. FAVRE AND V. GUEDJ, *Dynamique des applications rationnelles des espaces multiprojectifs*, *Indiana Univ. Math. J.* **50** (2001), no. 2, 881–934.
- [11] C.O. KISELMAN, *Attenuating the singularities of plurisubharmonic functions*, *Ann. Polon. Math.* **LX.2** (1994), 173–197.
- [12] A. RASHKOVSKII, *Plurisubharmonic functions with multicircled singularities*, *Visnyk Hark. Nats. Univ.* **475** (2000), 162–169.
- [13] A. RASHKOVSKII, *Lelong numbers with respect to regular plurisubharmonic weights*, *Results Math.* **39** (2001), 320–332.
- [14] A. RASHKOVSKII, *Singularities of plurisubharmonic functions and positive closed currents*, arXiv:0108159.

- [15] A. RASHKOVSKII, *Relative types and extremal problems for plurisubharmonic functions*, Int. Math. Res. Not. **2006** (2006), Article ID 76283, 26 p.
- [16] A. RASHKOVSKII, *Analytic approximations of plurisubharmonic singularities*, Math. Z. **275** (2013), Issue 3, 1217–1238.
- [17] B. TEISSIER, *Sur une inégalité à la Minkowski*, Annals Math. **106** (1977), no. 1, 38-44.
- [18] J. WIKLUND, *Pluricomplex charge at weak singularities*, arXiv:0510671.
- [19] V.P. ZAHARIUTA, *Spaces of analytic functions and Complex Potential Theory*, Linear Topological Spaces and Complex Analysis **1** (1994), 74–146.

Tek/nat, University of Stavanger, 4036 Stavanger, Norway  
alexander.rashkovskii@uis.no